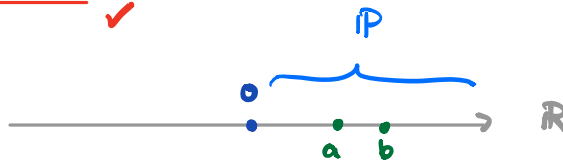


MATH 2050C Lecture 2 (Jan 12)

[[HW1 has been posted, due date Jan 20 (Fri).]]

Goal: \mathbb{R} is a complete ordered field.
today ✓

Ordering \rightsquigarrow \mathbb{R} as a real line



Defⁿ/Thm: $\exists \phi \neq \mathbb{P} := \{ \text{"positive" real numbers} \} \subseteq \mathbb{R}$ st.

(01): $a, b \in \mathbb{P} \Rightarrow a + b, ab \in \mathbb{P}$

(02): **Trichotomy**: $\forall a \in \mathbb{R}$, one and only one of the following holds:

$$a \in \mathbb{P} \quad \text{or} \quad a = 0 \quad \text{or} \quad -a \in \mathbb{P}$$

Notation: $a > 0$ if $a \in \mathbb{P}$; $a \geq 0$ if $a \in \mathbb{P} \cup \{0\}$

$a < 0$ if $-a \in \mathbb{P}$; $a \leq 0$ if $-a \in \mathbb{P} \cup \{0\}$

Define: $a > b$ if $a - b \in \mathbb{P}$

$a \geq b$ if $a - b \in \mathbb{P} \cup \{0\}$

Prop: (Rules of inequalities) Let $a, b, c \in \mathbb{R}$.

(a) $a > b$ and $b > c \Rightarrow a > c$

(b) $a > b \Rightarrow a + c > b + c$.

(c) $a > b \Rightarrow \begin{cases} ac > bc & \text{if } c > 0 \\ ac < bc & \text{if } c < 0 \end{cases}$

Proof: (a) By defⁿ, $a > b \Leftrightarrow a - b \in \mathbb{P}$
also $b > c \Leftrightarrow b - c \in \mathbb{P}$

By (01), $a - c = (a - b) + (b - c) \in \mathbb{P} \Rightarrow a > c$.
↑
(A2), (A3) \mathbb{P} \mathbb{P}
(A4)

(b) Exercise.

(c) By defⁿ, $a > b \Leftrightarrow a - b \in \mathbb{P}$.

Given $c > 0$, i.e. $c \in \mathbb{P}$, then by (O1)

$$ac - bc \stackrel{(O)}{=} \underbrace{(a-b)}_{\mathbb{P}} \cdot \underbrace{c}_{\mathbb{P}} \in \mathbb{P} \Rightarrow ac > bc.$$

Exercise for the case $c < 0$. _____ ◻

Thm 1: \mathbb{P} contains all natural numbers, i.e. $\mathbb{N} \subset \mathbb{P}$

Lemma: $a^2 \geq 0 \quad \forall a \in \mathbb{R}$.

Proof: By (O2), there are 3 possible cases:

Case 1: $a \in \mathbb{P}$

$$a^2 = \underbrace{a}_{\mathbb{P}} \cdot \underbrace{a}_{\mathbb{P}} \stackrel{(O2)}{\in} \mathbb{P} \quad \text{so } a^2 \geq 0.$$

Case 2: $a = 0$

$$a^2 = 0 \cdot 0 = 0 \quad \text{so } a^2 \geq 0.$$

Case 3: $-a \in \mathbb{P}$

$$a^2 \stackrel{\text{Ex.}}{=} (-a)^2 = \underbrace{(-a)}_{\mathbb{P}} \cdot \underbrace{(-a)}_{\mathbb{P}} \stackrel{(O2)}{\in} \mathbb{P} \quad \text{so } a^2 \geq 0. \quad \text{_____ } \circ$$

Proof of Thm 1: Use M.I. to show $n \in \mathbb{P} \quad \forall n \in \mathbb{N}$.

$$\underline{n=1}: \quad 1 = 1 \cdot 1 = 1^2 \stackrel{\text{Lemma}}{\geq} 0 \quad \text{and } 1 \neq 0 \quad (\text{by (M3)}).$$

So, $1 \in \mathbb{P}$

Assume $n=k$ is true, i.e. $k \in \mathbb{P}$.

Then $\underbrace{k}_{\mathbb{P}} + \underbrace{1}_{\mathbb{P}} \in \mathbb{P}$ by (O1), so $n=k+1$ is true. _____ ◻

Thm 2: $0 \leq a < \varepsilon \quad \forall \varepsilon > 0 \Rightarrow a = 0.$

(i.e. there is no "smallest" positive real number.)

Proof: By Contradiction. Suppose $a \neq 0$, then $a > 0$.

Note that $\frac{1}{2} > 0$ [why? If not, then $-\frac{1}{2} > 0$ ⁽⁰²⁾

$$\Rightarrow (-\frac{1}{2}) + (-\frac{1}{2}) = -1 > 0 \quad \text{[(01)]}$$

False $\because 1 > 0$

By (01), $\frac{1}{2} \cdot a \in \mathbb{P}$, ie. $\frac{1}{2}a > 0$.

False. (Ex: why?)

Choose $\varepsilon = \frac{1}{2}a > 0$, by assumption, $a < \frac{1}{2}a$ _____.

Prop: (1) $ab > 0 \Rightarrow$ either $a > 0$ and $b > 0$
or $a < 0$ and $b < 0$.

(2) $ab < 0 \Rightarrow$ either $a > 0$ and $b < 0$
or $a < 0$ and $b > 0$.

Pf: Exercise.